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Connections in Bergmann Manifolds

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This paper serves two purposes. On one hand it corrects an error made in an earlier paper and brought to my attention by a paper of Borowiec, who noted some problems with the "Hyperchristoffel connection." It is shown here that the "Hyperchristoffel connection" does not transform like a connection, nor is it metric, as I erroneously claimed. On the other hand, the hyperspin geometry is reformulated in terms of G-structures, also motivated by Borowiec's comment. This provides a mathematically precise, index-free formulation of hyperspinors and Bergmann manifolds, and in addition allows me to use the old results of Weyl-Cartan, corrected by Kobayashi *et al.*, about the existence of torsion-free G-connections. I find that in general a B_N, $N > 2$, does not possess a torsion-free G-connection.

1. INTRODUCTION

This paper is divided into two main parts. The first part, mainly contained in Section 2, corrects a mistake in my paper on the "Hyperchristoffel connection" (Holm, 1986), where I thought that I discovered a unique torsion-free and metric connection for the Finsler geometry of Bergmann manifolds. I called it "hyperchristoffel connection" because of its similarity to the Christoffel formula of Riemannian geometry. Borowiec (1988) pointed out some problems with the connection, namely that one particular identity was not fulfilled, and claimed that the "hyperchristoffel connection" did not transform like a connection. Though his findings were correct, the arguments he gave were not conclusive because he did not take into account the special properties of the hyperspin structure. He based his analysis solely on the chronometric g. Due to the complicated structure of g, there are many relations among its components, which have to be taken into

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account. This can be done by using the frame field σ as an independent dynamical tensor field, and looking upon g as a derived field. I will review his argumentation and present the correct way of looking at this problem.

The second part of this paper is mainly a reformulation of the hyperspin theory in terms of G-structures, as attempted by Borowiec. Borowiec used as group G the invariance group of the chronometric g, without specifying g or G. This is important because for a general N-ic form there might be no invariance group, and moreover, if there is, then it might not be the correct structure group. I will explicitly state the groups G which are used for defining a hyperspin structure and thereby hopefully remove any uncertainties about the structure. My definition reveals more clearly the mathematical structure of the theory by using fiber bundle formalism, it is an index-free formulation, and one can use known results in the theory of G-structures to show that for $N > 2$, a B_N does not possess in general a torsion-free G-connection.

What follows is only a short review of Bergmann manifolds, as details can be found in D. Finkelstein *et al.* (1986) and Holm (1988).

The idea of hyperspinors originated from an attempt to use spinors as the building blocks of time space, as it is done in the $SL_2 := SL(2, \mathrm{C})$ spinor calculus. This calculus has been known at least since van der Waerden's (1929) spinor analysis and has been extended by many authors; the best comprehensive source about the subject might be Penrose and Rindler (1984). Spinorial techniques have long been helpful in classical gravity, as, for instance, in the Petrov classification, the lens effect, or the positive energy theorem. Recently Ashtekar (1987) demonstrated with his new variables some advantages of the 2-spinor calculus in quantum gravity as well.

I take here the admittedly speculative perspective that the appearance of spinors is not accidental and a convenient calculational tool, but reflects some deeper properties of time space. The spinor structure may arise as an approximation of an underlying discrete network of quantum events (D. Finkelstein, 1989).

The SL_2 spinor calculus has a drawback; it works only if the dimension of the manifold is 2^2. In order to build higher-dimensional time spaces and still use a kind of spinor substructure we had to modify the concept of a spinor. The spinors we use are not spinors according to the usual definition, and lead to hyperspaces, so we call them hyperspinors. The Kaluza-Klein spaces built from hyperspinors do not form Riemannian geometries. This should not be disturbing, because after all we do not know if the world in higher dimensions is still Riemannian. In fact, if one looks at how one can generalize the $SO(3, 1)$ structure group (or Poincaré group) of ordinary space-time to higher dimensions, one finds that the hyperspin geometry is an extension as natural as Riemannian geometry is. Kaluza-Klein theories with non-Riemannian geometry have also been considered previously by Weinberg (1984), but for a different reason.

Hyperspinors are elements of the complex vector space $\mathbb{C}^N$, which transform under the fundamental representation of SL_N; antihyperspinors are elements of the complex conjugate space $\bar{\mathbb{C}}^{\dot{N}}$ and transform according to the complex conjugate representation of SL_N. A Bergmann manifold B_N is a sufficiently differentiable N^2-dimensional real manifold which admits hyperspinors. In addition, there exists the spin map σ, which is a real linear isomorphism between the tangent space of B_N (for a given basis) and the linear real space of sesquispinors $\mathbb{H}$, where $\mathbb{H}=\{\Psi\in\mathbb{C}^N\otimes\bar{\mathbb{C}}^{\dot{N}}\,|\,\Psi=\Psi^{\mathrm{H}}\}$. The superscript H stands for Hermitian conjugation. The sesquispinors can be represented by complex Hermitian $N\times N$ matrices $\Psi^{A\dot{A}}$, where $\dot{A}$ is independent of A, but transforms according to the complex conjugate representation $\bar{\Lambda}$ of SL_N. The transformation law is therefore $\Psi'=\Lambda\Psi\Lambda^{\mathrm{H}}$. Because of the Hermiticity condition on the elements of $\mathbb{H}$, the dimension of a B_N is fixed to be N^2.

The spin map, sometimes also called soldering form or Infeld–van der Waerden symbol, is the main dynamical variable of the theory. It is used to construct the *chronometric* g, an SL_N-invariant tensor field in the tangent space of B_N. One starts with an SL_N invariant (the determinant) in the sesquispinor space and maps it with σ into the tangent space:

$$g_{\alpha_1\ldots\alpha_N}=\frac{1}{(N-1)!}\,\varepsilon_{A_1\ldots A_N}\varepsilon_{\dot{A}_1\ldots\dot{A}_N}\sigma^{A_1\dot{A}_1}{}_{\alpha_1}\sigma^{A_2\dot{A}_2}{}_{\alpha_2}\ldots\sigma^{A_N\dot{A}_N}{}_{\alpha_N} \tag{1}$$

The small Greek indices belong to the tangent space and run over $1,\ldots,N^2$, whereas capital Latin ones are hyperspin indices and run over $1,\ldots,N$. Repeated indices are summed over. The tensor obtained in such a way is symmetric in all its indices. The dual chronometric $g^{\alpha_1\cdots\alpha_N}$ is obtained by simply using the inverse to σ and repeating the procedure in (1). The normalization is chosen such that

$$g_{\alpha\gamma_2\ldots\gamma_N}g^{\beta\gamma_2\ldots\gamma_N}=\delta_\alpha{}^\beta \tag{2}$$

For $N=2$ the chronometric is Einstein's metric field tensor $g_{\alpha\beta}$, and a B_2 is nothing else than a Lorentzian manifold with signature $(+,-,-,-)$. For $N>2$ the geometry is no longer Riemannian, but is known as Finslerian. In general the proper time $d\tau$ is a homogeneous function of the coordinates of degree N. With the notation $v^{A\dot{A}}:=\sigma_\alpha{}^{A\dot{A}}v^\alpha$ for a tangent vector v^α one can write the invariant line element as

$$d\tau^N=g_{\alpha_1\ldots\alpha_N}v^{\alpha_1}v^{\alpha_2}\ldots v^{\alpha_N}=N\,\det(v^{A\dot{A}})$$

Orthogonality is now a relation between N vectors, because g is an N-ic form. Another new aspect is that g no longer provides an isomorphism between the vector space and its dual. Lowering an index of a vector results in a covector with $N-1$ indices. The future light cone is defined to be the σ-image of the positive-definite sesquispinors, giving the hyperspin geometry a global causal structure.

Taking the spin map as a dynamical variable leads to a hypergravity theory. A natural candidate for a Lagrangian of hypergravity was introduced in D. Finkelstein *et al.* (1987) and S. Finkelstein (1988), which leads to interesting field equations and cosmological solutions (Holm, 1988). The spin map σ was taken as the sole dynamical variable, justified in part by the existence of the hyperchristoffel "connection".

2. THE HYPERCHRISTOFFEL "CONNECTION"

One of the first questions I tried to decide was if for the Finsler case there exists an analog of the Christoffel connection of Riemannian geometry, namely a torsion-free and chronometric connection. Chronometric means of course that the covariant derivative of g vanishes. In my paper (Holm, 1986) I thought that the answer to that question was affirmative. The "connection C" I found was

$$C_{\alpha\beta}{}^{\gamma} = \frac{1}{N} g^{\{\delta'\}\gamma}(\partial_{\alpha} g_{\beta\{\delta'\}} + \partial_{\beta} g_{\alpha\{\delta'\}} - \partial_{\delta} g_{\alpha\beta\{\delta''\}}) \tag{3}$$

I use here the notion of collective indices, where the kind of bracket stands for the symmetry, i.e., $\{\cdot\}$ stands for symmetrized, $[\cdot]$ for antisymmetrized indices. The number of indices is N; the number of primes tells how many indices are omitted, i.e., $\{\delta'\} = \delta_1 \ldots \delta_{N-1}$.

I used an "Ansatz" for $g_{\gamma\{\delta'\}} C_{\alpha\beta}{}^{\gamma} := [\alpha\beta, \delta_1 \ldots \delta_{N-1}]$ to solve the equation $Dg = 0$. Using the inverse chronometric, I could obtain $C_{\alpha\beta}{}^{\gamma}$, but did not verify that these coefficients were really connection coefficients, i.e., had the right transformation properties. Borowiec (1988) remarked that the $[\alpha\beta, \delta_1\delta_2]$ for $N=3$ did not transform into themselves (?) under gauge transformations, but did not quite specify which gauge transformations he had in mind.

I calculated the transformation law of C under coordinate transformations. Let us consider the transformation from a coordinate system x_{α} to a system with coordinates $x'_{\alpha'}$. Let X be the Jacobian matrix belonging to that transformation, i.e., $X^{\alpha}{}_{\beta'} := \partial x_{\alpha} / \partial x'_{\beta'}$. In this notation tensors transform as $g^{\alpha\beta} = g'^{\alpha'\beta'} X^{\alpha}{}_{\alpha'} X^{\beta}{}_{\beta'}$. I also use the short-hand notation $\partial_{\alpha'} X^{\gamma}{}_{\beta'} := X^{\gamma}{}_{\beta',\alpha'}$. For $N=3$ one obtains

$$\begin{aligned} C'_{\alpha'\beta'}{}^{\gamma'} = {} & X^{\alpha}{}_{\alpha'} X^{\beta}{}_{\beta'} X^{-1\gamma'}{}_{\gamma} C_{\alpha\beta}{}^{\gamma} + \tfrac{2}{3} X^{-1\gamma'}{}_{\tau} X^{\tau}{}_{\alpha',\beta'} + \tfrac{1}{3} g'^{\gamma'\tau'\sigma'} \\ & \times \{ g'_{\beta'\tau'\omega'} X^{-1\omega'}{}_{\nu} X^{\nu}{}_{\alpha',\sigma'} + g_{\alpha'\tau'\omega'} X^{-1\omega'}{}_{\nu} X^{\nu}{}_{\beta',\sigma'} - g_{\alpha'\beta'\omega'} X^{-1\omega'}{}_{\nu} X^{\nu}{}_{\tau',\sigma'} \} \end{aligned} \tag{4}$$

Apart from the factor $\frac{2}{3}$, the first two terms constitute the familiar transformation law of a connection. For (3) to be a connection, the last three terms have to add up to $\frac{1}{3}$ of the second term. That they do not is not so clear, because $g_{\alpha\beta\gamma}$ is derived from the spin map σ, and therefore not all its components are independent. Only for $N=2$ can any symmetric tensor $g_{\alpha\beta}$ with the Lorentz signature be derived from σ. For $N>2$ the set of all symmetric $g_{\{\alpha\}}$ derived from σ is only a small subset of the set of all totally symmetric tensors $t_{\{\alpha\}}$. For $N=2$, σ has 16 real variables minus the 6 coming from the local SL_2 invariance. This leaves 10 variables for the derived object $g_{\alpha\beta}$, which is exactly the same number of variables as for a symmetric tensor $t_{\alpha\beta}$ in 4 dimensions. Therefore g and σ can be used interchangeably; neither variable contains more information than the other. For $N=3$, σ has $81-16=65$ real variables, but a totally symmetric tensor $t_{\alpha\beta\gamma}$ in 9 dimensions has 165 free parameters. Necessarily there are many algebraic relations among the components of g. Therefore a more careful analysis, where all these relations can be taken into account, is needed in order to show that (4) is not the usual transformation law of a connection.

The problem of hidden identities can be solved if one uses σ instead of g in all calculations. This increases immediately the complexity of the equations, because instead of a linear equation in g, one has to deal with a cubic equation in σ (for $N=3$). Calculations are best performed in a determinantal chart, where σ is a simple delta function. In this chart the chronometric has the following form:

$$g_{A\dot{A}B\dot{B}C\dot{C}}=\tfrac{1}{2}\varepsilon_{ABC}\,\varepsilon_{\dot{A}\dot{B}\dot{C}}$$

It consists only of ones and zeros (apart from normalization). Even with this simplification calculations can hardly be done analytically. Equation (4) consists of 9^3 equations, and the examination is a task best suited for a symbolic calculation program like REDUCE (Hearn, 1987). Even with REDUCE, I could not check (4) in its full generality. Therefore I took a special coordinate transformation X, which deviated only in one coordinate from the identity map. With this simplification I could show that the last 4 terms of (4) really did not add up to the familiar $X^{-1\gamma'}{}_{\tau}X^{\tau}{}_{\alpha'\beta'}$. Because of this fact, I agree with Borowiec: C does not transform as a connection, and as such is just defined in one coordinate chart.

A second observation of Borowiec is that apparently $[\alpha\beta,\mu\nu]$ does not obey the identity

$$h^{\mu\nu}{}_{\tau\delta}[\alpha\beta,\mu\nu]=[\alpha\beta,\tau\delta] \tag{5}$$

The contraction tensor h is defined as $h^{\mu\nu}{}_{\tau\gamma}:=g^{\mu\nu\gamma}g_{\tau\delta\gamma}$ (Holm, 1988) and has the property of a projection operator (Borowiec, 1988). The identity has to hold because of the definition of $[\alpha\beta,\delta\tau]$ and (2). Checking (5), one finds that it implies

$$\tfrac{1}{6}(\partial_\gamma g_{\alpha\beta\delta}+\partial_\delta g_{\alpha\beta\gamma})-\tfrac{1}{3}(g_{\beta\sigma\tau}\partial_\alpha h^{\sigma\tau}{}_{\gamma\delta}+g_{\alpha\sigma\tau}\partial_\beta h^{\sigma\tau}{}_{\gamma\delta}-h^{\sigma\tau}{}_{\gamma\delta}\partial_\sigma g_{\tau\alpha\beta})=0 \qquad (6)$$

Again, the statement that this equation is not obeyed requires a rigorous proof by looking at the SL_3 substructure because of possible hidden identities in g. In order to keep the calculations manageable, I chose a concrete simple spin map that was constant in all but one coordinate. I used the form of the standard Hermitian basis of the Lie algebra of U_3 with a suitable normalization as a basis for σ. The spin map had a linear coordinate dependence only in one coordinate. I put equation (6) on the computer with REDUCE and obtained the result that (6) is false, and accordingly (5) is not fulfilled. While $g_{\alpha\beta\gamma}C^\gamma{}_{\mu\nu}$ is a solution to $Dg=0$, the inversion of the expression $[\alpha\beta, \mu\nu]$ to a connection $C^\gamma{}_{\alpha\beta}$ is not possible. Because (5) is not true, a repeated application of $g_{\mu\nu\gamma}$ to $C^\gamma{}_{\alpha\beta}$ does not yield the same expression for $[\alpha\beta, \mu\nu]$ one started with. This also implies that C is not even metric.

The only way in which C of (4) can be a connection is if the geometry of the B_N is so special that equations (4) and (5) are fulfilled. This means that there are many relations among the matrix elements of σ and its first derivatives. Although this cannot be excluded, it seems very unlikely. Moreover, it would put an unnecessary restriction on a B_N.

Because of the difficulty of performing local calculations even for $N=3$, I turn now to a different approach to be able to make a statement of more global nature about the existence of a torsion-free chronometric connection. It uses the theory of G-structures.

3. THE HYPERSPIN STRUCTURE

In this section a definition of a hyperspin structure will be given, which is a more abstract way of defining Bergmann manifolds in terms of G-structures. Bergmann manifolds are related to hyperspin structures in the same way as Riemannian manifolds are to $O(n)$-structures.

The use of G-structures has the advantage of introducing the terminology of the nowadays very common fiber bundle formalism. Moreover, I can use some already known results in the theory of G-connections. I will be able to show that in general a B_N, $N>2$, does not possess a torsion-free and chronometric connection.

First I will review briefly the general theory of G-structures; for details I recommend the books by Kobayashi (1972) and Sternberg (1964). Let M be a real differentiable manifold of dimension $n=N^2$. Let $L(M)$ denote the bundle of linear frames over M. The $L(M)$ is a principal fiber bundle over M with structure group $GL(n, R):= GL_n$, the general linear group.

Let G be a closed Lie subgroup of GL_n. A G-structure P_G is then a reduction of $L(M)$ to the subgroup G. In other words, P_G is a differentiable subbundle of $L(M)$ with structure group G. A G-connection is a connection in P_G.

I start with the construction of the hyperspin structure corresponding to B_2, because it is the manifold of classical general relativity. It is only a warmup exercise, because I know the outcome already, but it shows the way to generalize it to the case $N>2$.

The question to solve is how the structure group G for the tangent space of B_2 arises if I only know the structure group $\tilde{G}$ for the spinor space, in this case SL_2. The answer is to look at how the elements Ψ of $\mathbb{H}$ transform, because they correspond to real tangent vectors through σ. I recall that they transform under $\Lambda \in SL_2$ as $\Psi' = \Lambda\Psi\Lambda^{\mathrm{H}}$. This induces a real linear transformation on the tangent space of B_2. Because of the two-sided transformation behavior of Ψ, the elements of the center of SL_2, which is $Z_2 := \{1, -1\}$, get identified. This means the tangent space group G is SL_2/Z_2, which is a real subgroup of $GL(4, R)$, namely $SO_0(3, 1)$. Because the notion of a group G divided by its center Z will come up frequently, I will use a special notation for it. Let me define $PG := G/Z$. Therefore, a B_2 is a G-structure with $G = PSL_2$. The appearance of the proper orthochronous Lorentz group gives B_2 a 4-dimensional Riemannian geometry with metrical tensor of signature $(1, 3)$, a so-called Lorentz structure.

Because I want to work with SL_2 spinors, I also have to assume the existence of a spinor structure $P_{\tilde{G}}$ on M, which is a twofold covering of the bundle P_G. Note that $P_{\tilde{G}}$ is not a $\tilde{G}$-structure, but only a $\tilde{G}$-bundle. A necessary and sufficient condition for this cover to exist is the vanishing of a certain topological obstruction class. For an oriented and time-oriented Lorentzian manifold it is known (Bichteler, 1968)[2] that the obstruction class is equivalent to the second Stiefel–Whitney class of M.

The spin map σ of Section 1 is a vector bundle isomorphism from the tangent bundle $T(M)$ to $H(M)$, where $H(M)$ is a 4-dimensional vector bundle, whose typical fiber is the real linear space $\mathbb{H}$ defined previously. $\sigma^{A\dot{A}}{}_{\mu}$ is here the linear transformation from a holonomic frame dx^{μ} to an orthonormal frame $\sigma^{A\dot{A}}$ give by

$$\sigma^{A\dot{A}} = \sigma^{A\dot{A}}{}_{\mu}\, dx^{\mu} \tag{7}$$

On the level of principle bundles σ can be thought of as an R^4-valued one-form (the soldering form or 4-bein) on the PSL_2 bundle. It is this "soldering" that distinguishes the G-structure from other principle G-bundles and makes it a reduction of $L(M)$. In a way the spin map is kind of a Higgs field breaking the symmetry of GL_4 to PSL_2 (Trautmann, 1979).

[2]See also Baum (1981) for a more general theorem.

Of course, the field which breaks the symmetry is actually the chronometric g, but g can be derived from σ.

A G-connection on P_G is a linear connection due to the soldering. On the PSL_2 bundle there exists a unique connection D_0 which is characterized by the requirement to be torsion free. The D_0 is the well-known Levi-Civita connection of Riemannian geometry. The uniqueness of D_0 can be proven as well in the context of G-structures, as is done, for instance, in Sternberg (1964).

Because of the existence of the spinor structure, D_0 can be lifted to an SL_2 connection ∇_0, acting on Weyl spinors. This gives us then the unique spinor connection ∇_0. The metricity of D_0 and ∇_0 is inherent in the statement that they are PSL_2 and SL_2 connections, respectively. Metricity means that they satisfy

$$D_0 g = 0, \qquad \nabla_0 \varepsilon = 0 \tag{8}$$

where g is the PSL_2-invariant chronometric and ε is the antisymmetric SL_2 invariant Levi-Civita spinor.

The components of the connection with respect to a holonomic frame C and with respect to an orthonormal frame w are related by the usual transformation law of a connection:

$$w = \sigma C \sigma^{-1} + \sigma \, d\sigma^{-1} \tag{9}$$

After supplying this equation with indices and some rearranging, one obtains

$$\begin{aligned} D_\beta \sigma^{A\dot{A}}{}_\alpha &:= \partial_\beta \sigma^{A\dot{A}}{}_\alpha - \sigma^{A\dot{A}}{}_\gamma C^\gamma{}_{\alpha\beta} + w^{A\dot{A}}{}_{\dot{B}B\beta} \sigma^{B\dot{B}}{}_\alpha \\ &= \partial_\beta \sigma^{A\dot{A}}{}_\alpha - \sigma^{A\dot{A}}{}_\gamma C^\gamma{}_{\alpha\beta} + (\delta^{\dot{A}}{}_{\dot{B}} \Gamma_\beta{}^A{}_B + \delta^A{}_B \bar{\Gamma}_\beta{}^{\dot{A}}{}_{\dot{B}}) \sigma^{B\dot{B}}{}_\alpha = 0 \end{aligned} \tag{10}$$

The second equality used the lift of the vector connection w to a spinor connection Γ. This equation was called torque freeness in a previous paper (Holm, 1988), but from the point of view of G-structures it is a natural consequence.

The generalization to $N > 2$ is now an easy task because it involves only a change of the structure groups. The group acting on hyperspinors is SL_N, while the group acting on the sesquispinors is PSL_N, also known as the projective unimodular group. This comes about because a sesquispinor $h \in \mathbb{H}$ transforms under $\Lambda \in SL_N$ as $h \to \Lambda h \Lambda^{\mathrm{H}}$, so that the elements of the center of SL_N, which is Z_N, the cyclic group of order N, get identified. $\mathbb{H}$ is isomorphic to $\mathbb{R}^n$, $n = N^2$, and therefore the PSL_N transformation on $\mathbb{H}$ induces a real linear transformation on $\mathbb{R}^n$. This means that PSL_N has a real realization as a subgroup of GL_n. Then SL_N is an N-fold covering group of PSL_N and its fundamental representations are hyperspinors. The homomorphism of SL_N onto PSL_N can be made explicit in a similar way as for $N = 2$. Let the components of the isomorphism $\mathbb{H} \to \mathbb{R}^n$ be given in the form of a Hermitian basis $\sigma^{\mu}{}_{\dot{\Sigma}\Sigma}$, together with a dual basis $\sigma_{\mu}{}^{\Sigma\dot{\Sigma}}$, satisfying

$$\sigma^{\mu}{}_{\dot{\Sigma}\Sigma}\sigma_{\mu}{}^{\Omega\dot{\Omega}} = \delta^{\Omega}{}_{\Sigma}\delta^{\dot{\Omega}}{}_{\dot{\Sigma}} \qquad \text{and} \qquad \sigma^{\nu}{}_{\dot{\Sigma}\Sigma}\sigma_{\mu}{}^{\Sigma\dot{\Sigma}} = \delta^{\nu}{}_{\mu}$$

These matrices play the role of the Pauli spin forms together with the identity matrix. For each $\Lambda \in SL_N$ we then have an element $L(\Lambda) \in PSL_N$ given by

$$L^{\mu}{}_{\nu}(\Lambda) = \sigma_{\nu}{}^{\Sigma\dot{\Sigma}}\bar{\Lambda}^{\dot{\Omega}}{}_{\dot{\Sigma}}\Lambda^{\Omega}{}_{\Sigma}\sigma^{\mu}{}_{\dot{\Omega}\Omega} = \mathrm{Tr}\{\sigma_{\nu}\Lambda^{\mathrm{H}}\sigma^{\mu}\Lambda\}$$

It is straightforward to prove that $L(\Lambda_1) = L(\Lambda_2)$ if and only if Λ_1 and Λ_2 are related by $e^{i2\pi/a}\Lambda_1 = \Lambda_2$ for some a in $\{1, \ldots, N\}$. The kernel of this homomorphism consists of the elements of the form $e^{i2\pi/a}\mathbb{1}^{\Sigma}{}_{\Sigma'}$, where $\mathbb{1}^{\Sigma}{}_{\Sigma'}$ denotes the identity in SL_N, which is isomorphic to the group Z_N.

The last requirement for a hyperspin structure is that the structure group PSL_N can be lifted to SL_N, so that connections on vectors induce connections on hyperspinors. Having the group aspects settled, I can define a hyperspin structure in the following way.

Definition. A hyperspin structure $P_{\tilde{G}}$ is a $\tilde{G}$-bundle with $\tilde{G} = SL_N$ together with a strong bundle homomorphism η: $P_{\tilde{G}} \to P_G$, where P_G is a G-structure with $G = PSL_N$. Hyperspinors are sections in the associated bundle $P_{\tilde{G}} \times S$, where S is the representation space on which SL_N acts.

So far we always worked in the fundamental representation of SL_N so that $S = \mathbb{C}^N$. Equivalently I can introduce hyperspinors as equivariant functions Ψ: $P_{\tilde{G}} \to S$, where equivariant means that

$$\Psi(\tilde{g}\Lambda) = \mathfrak{R}(\Lambda^{-1})\Psi(\tilde{g})$$

$\mathfrak{R}$ is a representation of SL_N acting on S, $\tilde{g} \in P_{\tilde{G}}$, and $\Lambda \in SL_N$.

The bundle homomorphism η is sometimes also called the prolongation of the structure group to its central extension (Dabrowski, 1988). It is a strong bundle map, i.e., η induces the identity on M. The situation can be conveniently described by the following commutative diagram:

$$\begin{array}{ccc} P_{\tilde{G}} & \longleftarrow & P_{\tilde{G}} \times \tilde{G} \\ \eta \big\downarrow & & \big\downarrow \eta \times \rho \\ P_G & \longleftarrow & P_G \times G \\ \pi \big\downarrow & & \\ M & & \end{array}$$

Here ρ: $\tilde{G} \to G$ is a group homomorphism with discrete kernel, and the horizontal arrows denote the group action.

For a paracompact M a G-connection always exists (Kobayashi *et al.*, 1963). A PSL_N-connection D on P_G automatically satisfies

$$Dg = 0$$

By assumption, D can be lifted to a hyperspin connection ∇ on $P_{\tilde{G}}$. Then ∇ satisfies

$$\nabla \varepsilon = 0$$

Equations (9) and (10) are equally valid for arbitrary N.

The PSL_N structures are in a 1-1 correspondence with the cross sections $M \to L(M)/PSL_N$ (Kobayashi *et al.*, 1963). For $N=2$ it is well known that the cross sections can be described by (0, 2) tensor fields of signature (1-3). For a general N it is not clear if $L(M)/PSL_N$ is in a 1-1 correspondence with (0, N) tensor fields g which can be brought into the normal form (1). This statement would only be true if there is no bigger group than PSL_N that leaves such a g invariant. Therefore, one should focus the attention of a G-structure to its group G rather than to a particular G-invariant tensor field. N-ic forms do not characterize in general invariance groups as binary forms do. Our chronometric form g just does that because it was constructed that way. The frame field σ itself is a member of a coset in $L(M)/PSL_N$. Working with σ has the advantage that the constraints of the chronometric g are taken care of automatically. It is also clear from the construction that g and g^{-1} and tensors formed by contractions of them are the only PSL_N-invariant tensors that can be constructed in that particular representation.

A topological question which arises in the context of hyperspin structures is the following: What are the conditions on P_G which allow the existence of an N-fold covering bundle $P_{\tilde{G}}$? It turns out (Greub and Petry, 1978) that this is a condition on the transition functions $\tilde{f}_{ij}$: $U_{ij} \to P_{\tilde{G}}$, where U_i is a simple open covering of M, and $U_{ij} := U_i \cap U_j$. From the transition functions one can construct a cocycle p, which in turn characterizes an element of the second Čech cohomology group $H^2(M, Z_N)$. The vanishing of this obstruction class gives a necessary and sufficient condition for the existence of a prolongation $P_{\tilde{G}} \to P_G$. A further investigation of this obstruction class is under way (Holm, 1989). Another interesting aspect is that the prolongation is not necessarily unique, so that there might be several inequivalent prolongations leading to inequivalent hyperspin structures.

At the end of this section I add a generalization of a hyperspin structure to a conformal hyperspin structure which is a simple generalization of the procedure one does in the case of 4 dimensions.

Definition. A conformal hyperspin structure $CP_{\tilde{G}}$ is a $\tilde{G}$-bundle with $\tilde{G} = GL(N, C) := GL_N$ together with a strong bundle homomorphism $\eta : CP_{\tilde{G}} \to CP_G$, where CP_G is a G-structure with $G = GL_N/S^1$.

A conformal hyperspin structure may turn out to be useful in calculations on the hyperspin level, because by making the hyperspin connection on $CP_{\tilde{G}}$ traceless, one obtains a connection on $P_{\tilde{G}}$. The name conformal is justified because they transform the chronometric to a positive multiple of itself. In bundle coordinates one has

$$\varepsilon_{A_1\ldots A_N}\varepsilon_{\dot{A}_1\ldots\dot{A}_N}\Lambda^{A_1}{}_{B_1}\bar{\Lambda}^{\dot{A}_1}{}_{\dot{B}_1}\ldots\Lambda^{A_N}{}_{B_N}\bar{\Lambda}^{\dot{A}_N}{}_{\dot{B}_N}=c\varepsilon_{B_1\ldots B_N}\varepsilon_{\dot{B}_1\ldots\dot{B}_N}$$

where $c\in R^+$, $\Lambda\in GL_N/S^1$.

4. WEYL–CARTAN THEOREM

In this section I will apply the Weyl–Cartan theorem in order to show that *in general* a B_N does not possess a torsion-free connection. The first of the two theorems is in this form due to Kobayashi and Nagano (1965):

Theorem 1. Fix a Lie subgroup G of $GL(n, R)$, $n\geq 3$, and an n-dimensional manifold M which admits a G-structure. Then the following two statements are equivalent:

(i) Every G-structure P on M admits a torsion-free connection.
(ii) The Lie algebra g of G is one of the following:

$$gl(V),\ sl(V),\ co(V),\ o(V),\ gl(V,W),\ gl(V,W,c)\qquad\text{with }\dim W=1$$

In this notation V stands for the real vector space $\mathbb{R}^n$, and gl, sl, co, and o are the Lie algebras of the general linear, the special linear, the conformal, and the orthogonal group, respectively. For a subspace W of V, $gl(V, W)$ denotes the Lie algebras of linear transformations of V, leaving W invariant. For dim $W=1$ and a real number c, $gl(V, W, c)$ is defined as the subalgebra of $gl(V, W)$ consisting of matrices of the form

$$\begin{pmatrix} c\operatorname{Tr}A & b\\ 0 & A\end{pmatrix},\qquad A\in gl(n-1,R),\quad b\in R^{n-1}$$

For my purpose it is only important to notice that all these Lie algebras have $\dim g\geq\frac{1}{2}n(n-1)$.

The second theorem I will need is the Weyl–Cartan theorem, which in this form is due to Klingenberg (1959).

Theorem 2. If G and M are defined as before, then every G-structure P on M admits a unique torsion-free connection if and only if the Lie algebra of G is $o(V)$.

Both theorems are proved in this form in Kobayashi and Nagano (1965) by using the method of prolongations of Lie algebras. I do not want to repeat their proofs, but instead show how the dimensional constraint on the Lie algebra g of G arises. For that purpose, I examine Cartan's first structure equation:

$$d\sigma^i = w^i{}_k \wedge \sigma^k + T^i \tag{11}$$

where σ^i is a local frame, a one-form on M, $w^i{}_k$ is the connection one-form on M with values in g, and T^i is the torsion two-form on M. Because $d\sigma$ and T are two-forms, they can be thought of as tensors in $V \otimes \Lambda^2 V^*$, where w is a map of $V \to g$ and has components in $g \otimes V^*$. We ask now, when can (11) have a solution w with $T = 0$. Cartan's equation (11) is an algebraic system of $\frac{1}{2}n^2(n-1)$ equations for $n \cdot d$ unknowns, where $d := \dim g$. The left-hand side of (11) can take on arbitrary values by change of the local section. This implies that (11) has a solution only for $d \geq \frac{1}{2}n(n-1)$. The solution is unique if $d = \frac{1}{2}n(n-1)$.

This constraint on the dimensions of g has immediate implications for the existence of a torsion-free connection on P_G. The structure group of a B_N is PSL_N, and its Lie algebra has the dimension $\dim g = 2(N^2 - 1) = 2(n-1)$. Because $2(n-1) \leq \frac{1}{2}n(n-1)$ with the equality only valid for $n = 4$, I know that for $N > 2$, $sl(N)$ does not belong to the class of Lie algebras listed in Theorem 1. Therefore, I come to the important conclusion that in general a hyperspin structure does not possess a torsion-free connection. Moreover, according to Theorem 2, a unique torsion-free connection exists if and only if $g = o(n)$, which means that the geometry is (pseudo-) Riemannian, and the unique torsion-free connection in this case is the Levi-Civita connection. The existence of a unique torsion-free connection is therefore a special feature of Riemannian geometry. It is impossible to find a Christoffel-like formula for arbitrary Bergmann manifolds of $N > 2$.

5. CONCLUSIONS

Thanks to the comment by Borowiec, I was led to investigate the hyperchristoffel "connection" again. I could verify that this "connection" really was not a connection because it did not have the right transformation properties. Defining a hyperspin structure as a N-fold covering bundle of a special kind of G-structure, I could use some known results about the existence of torsion-free G-connections. The main result is that in general a B_N does not possess a torsion-free connection. Of course, this does not rule out the possibility that some manifolds might allow such a connection. If the manifold is special in some sense, it might be possible to find one. A trivial example is the flat manifold $\mathbb{R}^n$, which has a trivial torsion-free connection. Another example is provided by the group manifolds formed by the covering group of $U_N := U(N, C)$ (Holm, 1988). Also, there is a torsion-free G-connection that exists. Incidentally, it is the same connection which respects the Riemannian structure on that group.

It is also not ruled out that one can find some conditions on the torsion form that allow one to construct a unique connection, as is done, for example, for G-structures where $G = U_n$ (Klingenberg, 1959). There the vanishing of the mixed terms of the torsion form is a sufficient and necessary condition for the existence of a unique connection.

The most likely alternative seems to be to take torsion in higher-dimensional time space very seriously. If one cannot get rid of torsion, then one should look for its physical implications, This is not necessarily a weakness in theory of hyperspinors, but merely an unexpected physical consequence.

The existence of torsion will make it impossible to treat σ as a sole dynamical variable. Therefore one should derive the hypergravity field equations of D. Finkelstein *et al.* (1987) treating the connection as another free variable.

NOTE ADDED IN PROOF

I recently learned about the comment of H. Urbantke (1989) who also noted that one can use the results of Weyl-Cartan to show the nonexistence of a torsion-free chronometric connection for Bergmann manifolds of $N > 2$. But Urbantke also does not specify the group of hyperspin manifolds, so my paper hopefully will remove any prevailing uncertainties about the correct definition of a hyperspin structure.

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